CS 369: Introduction to Robotics

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Outline for today

- Kalman filter

Kalman filter

- Tractable implementation of the Bayes filter for continuous space
- Assumes:
  - Gaussian distributions
  - Linear dynamic system:
    - $x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$
    - $z_t = C_t x_t + \delta_t$

Components of a Kalman filter

A_t Matrix (*nxn*) that describes how the state evolves from *t-1* to *t* without controls or noise

B_t Matrix (*nxm*) that describes how the control u_t changes the state from *t-1* to *t*

C_t Matrix (*kxn*) that describes how to map the state x_t to an observation z_t

ε_t δ_t Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t, respectively

Kalman filter: initialization

Initial belief is normally distributed

$$bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

mean (of μ_0), covariance (of Σ_0)

Kalman filter: dynamics

State is linear function of previous state and control plus noise

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right)$$

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

Kalman filter: dynamics

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Kalman filter: observations

Observation is linear function of state plus noise

$$z_t = C_t x_t + \delta_t$$

$$p(z_t \mid x_t) = N\left(z_t; C_t x_t, Q_t\right)$$

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, Q_t\right) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

Kalman filter: observations

$$bel(x_t) = \quad \eta \quad p(z_t \mid x_t) \qquad\qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\sim N(z_t; C_t x_t, Q_t) \qquad \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Kalman filter

Algorithm Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$$

$$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$$

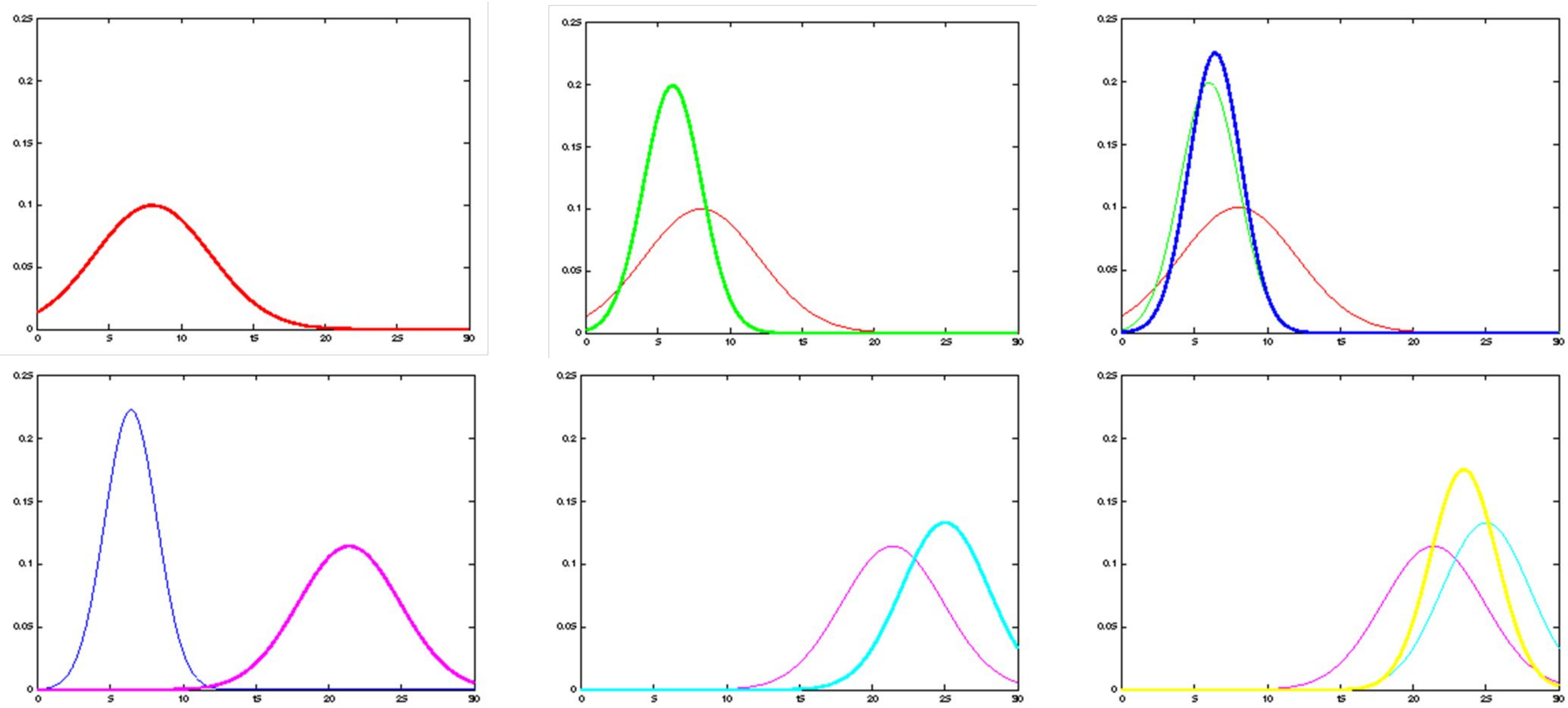
$$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$$

return μ_t, Σ_t

Probabilistic Robotics

Kalman filter illustration

Probabilistic Robotics



Kalman filter

- Highly efficient: polynomial in measurement dimensionality *k* and state dimensionality *n*

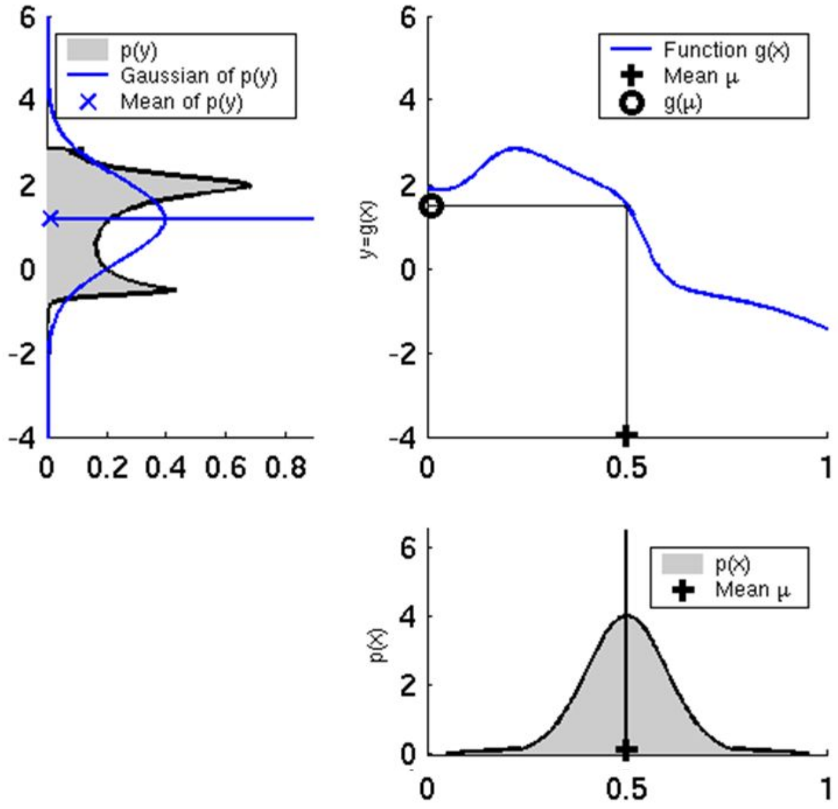
- Optimal for linear Gaussian systems

- Most robotics systems are nonlinear

$$x_t = g(u_t, x_{t-1}) + \varepsilon_t$$

$$z_t = h(x_t) + \delta_t$$

Nonlinear function



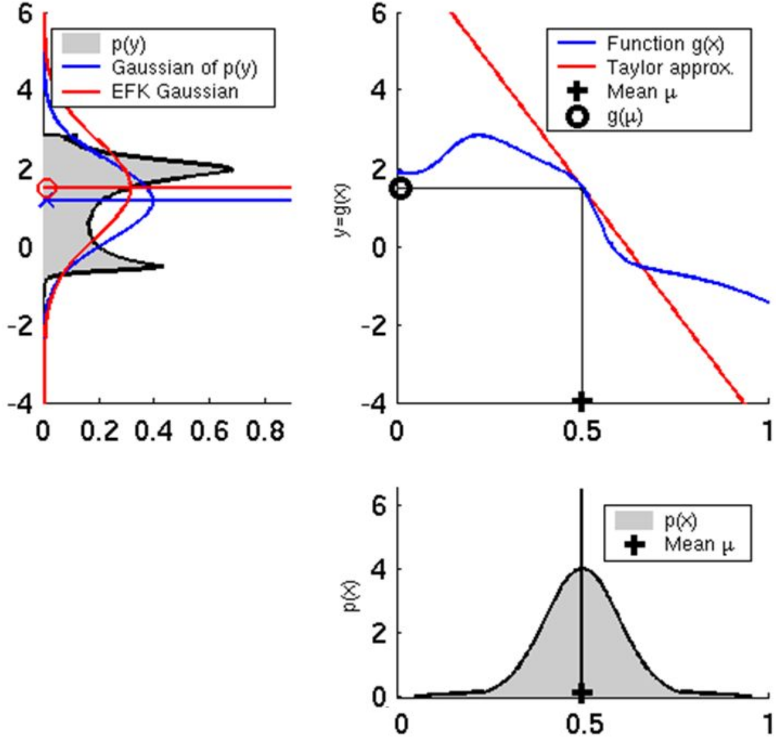
First-order Taylor expansion

- $$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

  $$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

- $$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

  $$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$



Extended Kalman filter (EKF)

Algorithm Extended_Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)**:**

$\bar{\mu}_t = g(u_t, \mu_{t-1})$

$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Probabilistic Robotics

EKF

- Highly efficient: polynomial in measurement dimensionality k and state dimensionality n
- Not optimal
- Can diverge if nonlinearities are large
- Works surprisingly well even when all assumptions are violated

Multi-hypothesis tracking

- Belief is represented by multiple hypotheses
- Each hypothesis is tracked by a Kalman filter

- Additional problems:
    - Data association: Which observation corresponds to which hypothesis?
    - Hypothesis management: When to add / delete hypotheses?

- Huge body of literature on target tracking, motion correspondence, etc.